Towards First-Order Linear Quantitative Logic

Matteo Capucci

University of Strathclyde / Independent Researcher (ARIA Creator)

in collaboration with **E. Komendantskaya**, **J. Bailitis**, **M. Spadetto**

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Advantages:

- quantitative **sequents** can axiomatize harmonic/addition as additive connectives,
- quantitative **satisfaction** accounts e.g. for property-driven training in NNV.

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The big picture

Bayesian probability, information theory,
machine learning, statistics, etc.

To give logical foundations to quantitative disciplines
***as they are*, not reinventing them.**

Why first-order?

We need a theory of **quantitative structures** and **quantitative satisfiability** for...

- NNV: to **fully interpret a neurosymbolic loss**:

$$\forall(i :_0 [1, N]).\ \Big(\big(\forall(j :_0 [1, k]).\ \hat{y}_i(j) \multimap f_\theta(\hat{x}_i j)\big)\ \wedge\ \lambda \cdot \forall(x :_\infty P(\hat{x}_i))\,.\,Q(f_\theta(x), \hat{y}_i)\Big)$$

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- host **modal logics**, or at least their Kripke semantics,

- do **structural mathematics**, chiefly category theory.

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types probability spaces $(X, \mathrm{d}x)$

predicates measurable $\varphi : X \to [0, \infty]$

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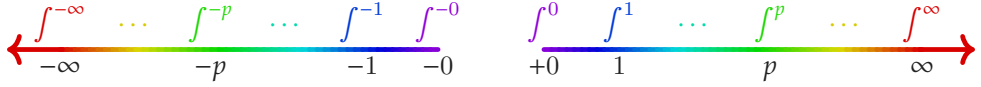
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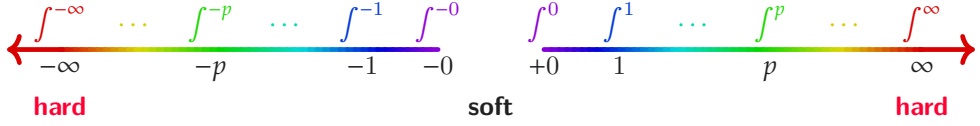
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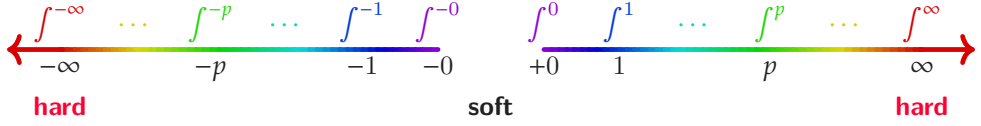
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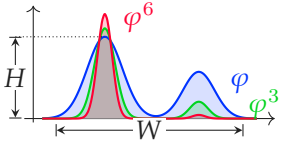
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- Heuristic: trading 'democracy' and 'autocracy', $\int^p \varphi \approx HW^{1/p}$



Integrals as quantifiers

Good logico-algebraic properties:

- **representability**: $\left(\int_y^p \varphi(x,y)\,\mathrm{d}y\right) \multimap \psi(x) = \int_y^{-p} \big(\varphi(x,y) \multimap \psi(x)\big)\,\mathrm{d}y$

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- **Frobenius**: $\psi(x) \otimes \int_y^p \varphi(x,y)\,\mathrm{d}y = \int_y^p \big(\psi(x) \otimes \varphi(x,y)\big)\,\mathrm{d}y$

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Entailment

From **(representability)** above we evince the definition of semantic entailment:

$$(\varphi \sqsubseteq_p \psi) := \int_{x\in X}^{-p} \varphi(x) \multimap \psi(x)\,\mathrm{d}x = [\![\forall^p x.\, \varphi(x) \multimap \psi(x)]\!] \qquad \text{recall } a \multimap b = b/a$$

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Aside from a formal route, this entailment is also motivated at the extremes:

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- Over $\{\varphi\} := (X, \varphi \cdot \mathrm{d}x)$, i.e. the measure $\mathrm{d}x$ reweighted by the density φ:

$$-\log(\varphi \sqsubseteq_p \psi) = D_p^{\mathrm{d}x}(\varphi \parallel \psi) \qquad \text{(relative } p\text{-entropy)}$$

where $D_0^{\mathrm{d}x}$ is KL divergence.[2]

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The above ideas are 3 years old, from (Capucci 2024), there we also observed that

reflexivity	$1 \leq (\varphi \sqsubseteq_p \varphi)$	**holds**
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Thus we have:

graded transitivity	$(\varphi \sqsubseteq_p \psi) \otimes (\psi \sqsubseteq_q \chi) \leq (\varphi \sqsubseteq_{p \oplus^* q} \chi)$	**holds!**

This makes the functions on X a **locally graded enriched preorder**:

grades in $([0,\infty],\oplus^*,\infty)$, values in $([0,\infty],\otimes,1)$

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Similar but distinct from "Bunched Fuzz" (wunder, Amorim, Baillot, and Gaboardi 2023)

A doctrine with graded enriched fibres

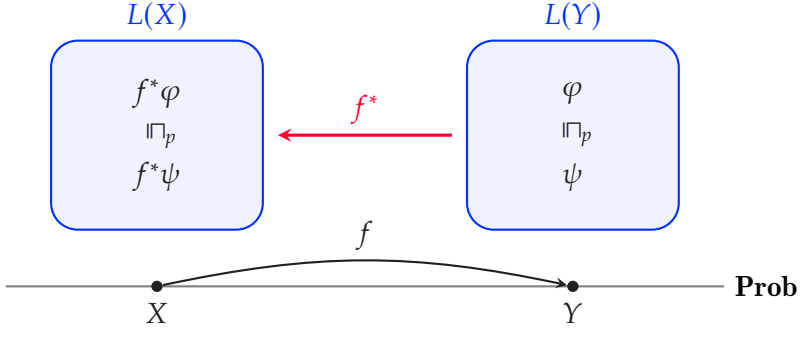
$$\mathbf{Prob}^{\mathsf{op}} \xrightarrow{L} (\oplus^*, \otimes)\text{-}\mathbf{Prd}$$

$$X \longmapsto L(X) = \big(\{X \to [0,\infty] \text{ msbl}\}, \sqsubseteq_{(-)}\big)$$



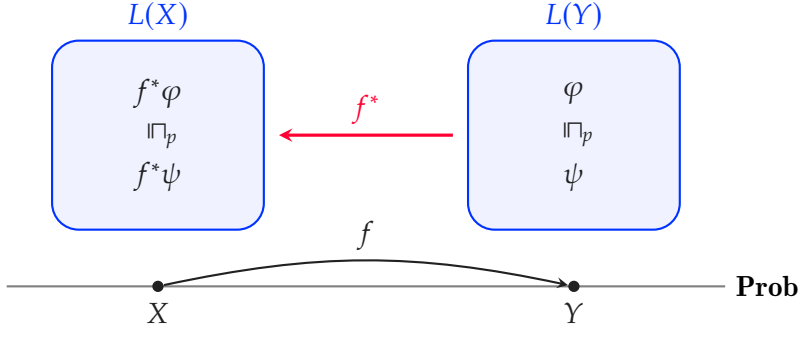
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Fibres carry, roughly, **the structure of a softale**, given by pointwise $\otimes$, $\oplus^p$, and $(-)^*$.

Soft quantifiers are “graded” adjoints

For a projection $\pi_X : X \times Y \to X$:

$$\bigvee \dashv \pi_X^* \dashv \bigwedge$$

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$$\begin{aligned} (\textstyle\int^{p}_{\pi_X} \varphi \sqsubseteq_p \psi) &= (\varphi \sqsubseteq_p \pi_X^* \psi) \\ (\pi_X^* \psi \sqsubseteq_p \varphi) &= (\psi \sqsubseteq_p \textstyle\int^{-p}_{\pi_X} \varphi) \end{aligned} \quad \xrightarrow{p \to +\infty} \quad \begin{aligned} \textstyle\bigvee_{\pi_X} \varphi \le \psi &\iff \varphi \le \pi_X^* \psi \\ \pi_X^* \psi \le \varphi &\iff \psi \le \textstyle\bigwedge_{\pi_X} \varphi \end{aligned}$$

The proof is by construction of L:

$$\begin{aligned} (\textstyle\int^{p}_{\pi_X} \varphi \sqsubseteq_p \psi) &= \int_{x \in X}^{-p} \Big(\int_{y \in Y}^{p} \varphi(x,y)\,\mathrm{d}y \Big) \multimap \psi(x)\,\mathrm{d}x \\ &= \int_{x \in X}^{-p} \int_{y \in Y}^{-p} \varphi(x,y) \multimap \psi(x)\,\mathrm{d}y\,\mathrm{d}x \qquad \text{(representability of } p\text{-integrals)} \\ &= \int_{(x,y) \in X \times Y}^{-p} \varphi(x,y) \multimap \psi(x)\,\mathrm{d}(x,y) = (\varphi \sqsubseteq_p \pi_X^* \psi) \qquad \text{(Fubini)} \end{aligned}$$

Beck–Chevalley and Frobenius hold too: L is a **"soft first-order" hyperdoctrine**.

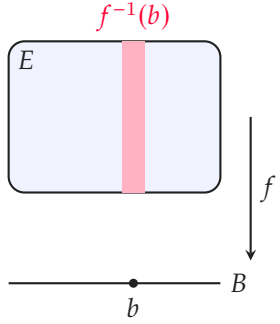
Soft quantifiers along non-singular maps

Actually, we have soft quantifiers on a much larger class of maps:

Theorem. Let $f : E \longrightarrow B$ be measurable, with B σ-finite, and $0 < p < \infty$. Then f^* has a left p-adjoint $\int_f^p \dashv_p f^*$ **iff** f is **non-singular** (preimages of null sets are null), and then

Radon–Nikodym derivative of the fiberwise mass of φ^p

$$\int_f^p \varphi = \left(\frac{\mathrm{d} f_*(\varphi^p \cdot \mathrm{d}e)}{\mathrm{d}b} \right)^{1/p}$$



the mass of φ^p on the fibres, as a density over B

Instances:

- projections $\pi_X : X \times Y \longrightarrow X$: the p-mean $\int_{y \in Y}^p \varphi(x, y)\, \mathrm{d}y$ again,

$p = 0$ is not covered by the theorem.

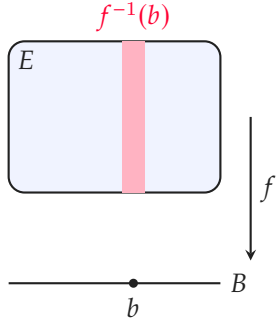
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the mass of φ^p on the fibres, as a density over B

Instances:

- projections $\pi_X : X \times Y \longrightarrow X$: the p-mean $\int_{y \in Y}^p \varphi(x, y)\, \mathrm{d}y$ again,
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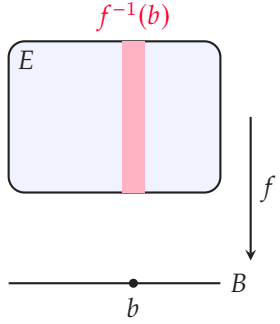
Soft quantifiers along non-singular maps

Actually, we have soft quantifiers on a much larger class of maps:

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Equality and entropy

On a **purely atomic** space $(X, \mathrm{d}x)$ the diagonal $\Delta : X \to X \times X$ is non-singular, so we can use the above to get **Lawvere's equality** (i.e. $(=) := \exists_\Delta(\top)$):

$$(x_1 =_p x_2) := (\textstyle\int_\Delta^p 1)(x_1, x_2) = \begin{cases} \mathrm{d}x\{x_1\}^{-1/p} & x_1 = x_2 \\ 0 & \text{otherwise} \end{cases} \xrightarrow{p\to\infty} \text{indicator of the diagonal}$$

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A **diffuse** space (e.g. Lebesgue $[0, 1]$) has a singular diagonal, so **it doesn't support equality** in this model.

Beyond locally graded hyperdoctrines

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the motivating sentence itself is **not expressible**!

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Solution: **graded, ordered contexts** $\quad x :_P X \quad \equiv \quad x_1 :_{p_1} A_1;\ \ldots;\ x_n :_{p_n} A_n$

$$L(X) := L(A_1 \times \cdots \times A_n), \qquad L(f) = f^* : L(B_1 \times \cdots \times B_n) \to L(A_1 \times \cdots \times A_n),$$

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$$=: \int_{x \in X}^{-P} \varphi(x) \multimap \psi(x)\, \mathrm{d}x \qquad \text{the } \textbf{mixed } P\textbf{-mean} \text{ of Benedek and Panzone (1961)}$$

Grading with context-length tuples appears already in Petricek, Orchard, and Mycroft (2014);

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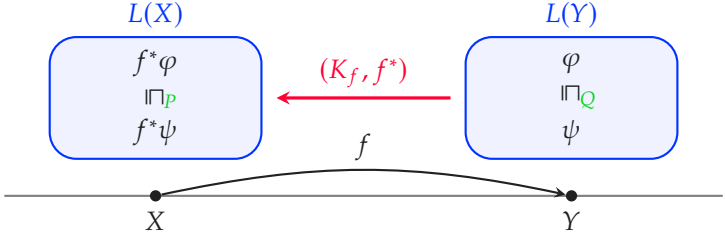
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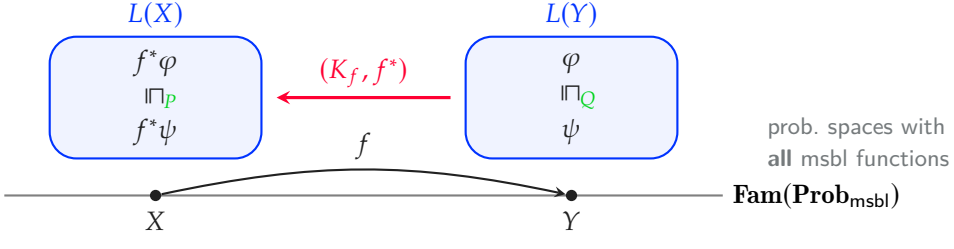
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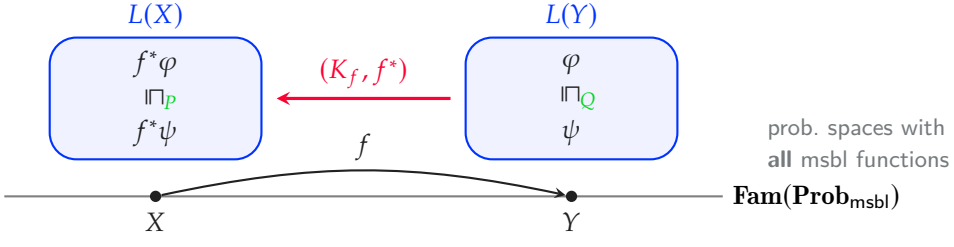
validity of a 'substitution' judgment
$= \left\| f^* : L^Q(Y) \to L^P(X) \right\|^{-1}$

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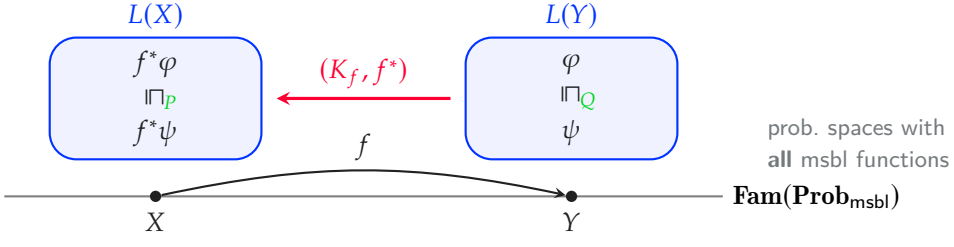
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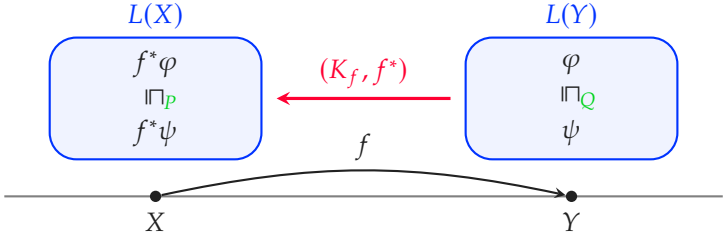
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$\mathbf{Fam}(\mathbf{FinMeas}_{\mathrm{msbl}})$

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A look at the sequent calculus

Just like QLL is moulded around MALL, QPL follows graded systems such as λR of Wood and Atkey (2022) and the *coeffect calculi* of Petricek, Orchard, and Mycroft (2014), Orchard, Liepelt, and Eades III (2019).

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environments: reified substitutions

from Wood and Atkey (2022)

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Notably, for the VAR term:

$T_{\mathrm{id}_A}(1, \infty) = 1/|A|$, the inverse **mass**, and $T_{\mathrm{id}_A}(\infty, 1) = \gamma_A$, the **grain**

(the smallest positive measure of a set)

Example: a finite set

$A = \{1, \ldots, n\}$, two measures: **counting** (mass n, grain 1) and **uniform** (mass 1, grain $1/n$).

	judgment	**counting**	**uniform**
type	A type	$1/n$	1
variable (term)	$x :_p A \vdash x :_q A$	$p \cdot (1/n)$	$q \cdot (1/n)$
identity (environment)	$x :_p A \Rightarrow x :_p A$	1	1
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softening costs in one model, **hardening** in the other

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discarding costs $p \cdot (A \text{ type})$: free in the uniform model, and for hard variables

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grade residual $q \leq^* p = 1/(1/p - 1/q)^+$, with $0 \leq^* 0 = \infty$

The calculus: types and contexts

Validities of types (base types C come with v_C in the signature):

$$\frac{v_C}{C \text{ type}} \qquad \frac{1}{\bullet \text{ type}}\,\bullet \qquad \frac{A \text{ type} \quad \otimes \quad B \text{ type}}{A \times B \text{ type}}\,\times$$

$$\frac{\infty}{\circ \text{ type}}\,\varnothing \qquad \frac{\kappa^* \otimes (A \text{ type})}{\kappa A \text{ type}}\,\kappa \qquad \frac{A \text{ type} \quad \oplus^* \quad B \text{ type}}{A + B \text{ type}}\,+$$

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Contexts:

$$\frac{1}{() \text{ ctx}}\ \text{EMPTY} \qquad \frac{x :_p X \text{ ctx} \quad \otimes \quad q \cdot (A \text{ type})}{x :_p X;\ y :_q A \text{ ctx}}\ \text{CTX}$$

since $\infty \cdot a = 1$, **a hard variable is free**.

The calculus: terms

Variables:

$$\frac{v_{C,p,q}}{x :_p C \vdash x :_q C}\ C\text{VAR} \qquad \frac{1}{x :_p \bullet \vdash x :_q \bullet}\ \bullet\text{VAR}$$

The variable rule is **not an axiom of validity** 1: it is "η-expanded".

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Promotion multiplies all grades by r and acts by r on the validity:

$$\frac{r \cdot (x :_P X \vdash t :_q A)}{x :_{r \otimes^* P} X \vdash t :_{r \otimes^* q} A}\ \mathsf{PR}$$

...plus all the introduction rules for the type formers (tbd)

The calculus: environments

$$\frac{1}{x :_P X \Rightarrow x :_P X}\ \mathsf{ID} \qquad \frac{x :_P X\ \mathsf{ctx}}{x :_P X \Rightarrow ()}\ \mathsf{WKN} \qquad \frac{x :_P X \Rightarrow s :_R Y \quad \otimes \quad x :_Q X \vdash t :_s A}{x :_{P \oplus^* Q} X \Rightarrow s :_R Y;\ t :_s A}\ \mathsf{EXT}$$

and more (tbd)

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EXT applied to an identity-like environment and a variable term gives the **diagonal**:

$$\frac{x :_{p_1} A \Rightarrow x :_{q_1} A \quad \otimes \quad x :_{p_2} A \vdash x :_{q_2} A}{x :_{p_1 \oplus^* p_2} A \Rightarrow x :_{q_1} A;\ x :_{q_2} A}\ \text{DIAG}$$

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Theorem. On atomic probability spaces, $K_\Delta(p, (q_1, q_2)) = \max_{p_1 \oplus^* p_2 = p} K_{\mathrm{id}}(p_1, q_1) \otimes T_{\mathrm{id}}(p_2, q_2)$: DIAG is **sound and complete** for the diagonal.

The calculus: substitution

Environments act on sequents and terms, and the validity of the **substitution** is a factor of the result:

$$\frac{x :_P X \Rightarrow t :_Q Y \quad \otimes \quad y :_Q Y \mid \Gamma \vdash \Delta}{x :_P X \mid \Gamma[y/t(x)] \vdash \Delta[y/t(x)]}\ \text{SUBST}$$

$$\frac{x :_P X \Rightarrow s :_Q Y \quad \otimes \quad y :_Q Y \vdash t :_q A}{x :_P X \vdash t[y/s(x)] :_q A}\ \text{T-SUBST}$$

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Lipschitz functoriality of t^*:

$$K_t(P,Q) \otimes (\varphi \sqsubseteq_Q \psi) \ \leq \ (t^*\varphi \sqsubseteq_P t^*\psi)$$

$$\dfrac{x :_P X \Rightarrow s :_Q Y \quad \otimes \quad y :_Q Y \vdash t :_q A}{x :_P X \vdash t[y/s(x)] :_q A}\ \text{T-SUBST}$$

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$$\dfrac{x :_P X \Rightarrow s :_Q Y \quad \otimes \quad y :_Q Y \vdash t :_q A}{x :_P X \vdash t[y/s(x)] :_q A}\ \text{T-SUBST}$$

This is how the **structural rules for variables** arise.

The calculus: structural rules

$$\underbrace{\frac{x :_P X \text{ ctx}}{x :_P X \mid A \vdash A}\ \mathsf{AX}}$$

graded reflexivity:

$$\underbrace{[\![x :_P X \text{ ctx}]\!]}_{=1 \text{ for prob. spaces}} \leq (\varphi \sqsubseteq_P \varphi)$$

$$\underbrace{\frac{x :_P X \mid \Gamma \vdash A, \Delta \quad \otimes \quad x :_Q X \mid \Gamma', A \vdash \Delta'}{x :_{P \oplus^* Q} X \mid \Gamma, \Gamma' \vdash \Delta, \Delta'}\ \mathsf{CUT}}$$

graded transitivity:

$$(\varphi \sqsubseteq_P \psi) \otimes (\psi \sqsubseteq_Q \chi) \leq (\varphi \sqsubseteq_{P \oplus^* Q} \chi)$$

$$\frac{x :_P X \text{ ctx}}{x :_P X \mid \vdash}\ \mathsf{EMP} \qquad \frac{x :_P X \mid \Gamma \vdash \Delta \quad \otimes \quad x :_Q X \mid \Gamma' \vdash \Delta'}{x :_{P \oplus^* Q} X \mid \Gamma, \Gamma' \vdash \Delta, \Delta'}\ \mathsf{MIX} \qquad \frac{0}{x :_P X \mid \Gamma \vdash \Delta}\ \mathsf{EFQ}$$

$$\frac{x :_P X \mid \Gamma \vdash A, \Delta}{x :_P X \mid \Gamma, A^\perp \vdash \Delta}\ (-)^\perp_{\mathsf{L}} \qquad \frac{x :_P X \mid \Gamma, A \vdash \Delta}{x :_P X \mid \Gamma \vdash A^\perp, \Delta}\ (-)^\perp_{\mathsf{R}}$$

The calculus: quantifiers

$$\frac{x :_P X;\ y :_Q Y \mid \Gamma, A \vdash \Delta}{x :_P X \mid \Gamma, \exists(y :_Q Y).\, A \vdash \Delta}\ \exists_{\mathsf{L}} \qquad \frac{x :_P X;\ y :_Q Y \mid \Gamma \vdash A, \Delta}{x :_P X \mid \Gamma \vdash \forall(y :_Q Y).\, A, \Delta}\ \forall_{\mathsf{R}}$$

$$\frac{x :_P X \vdash t :_q A \quad \otimes \quad x :_Q X \mid \Gamma \vdash \varphi[t/y], \Delta}{x :_{P \oplus^* Q} X \mid \Gamma \vdash \exists(y :_q A).\, \varphi, \Delta}\ \exists_{\mathsf{R}}$$

$$\frac{x :_P X \vdash t :_q A \quad \otimes \quad x :_Q X \mid \Gamma, \varphi[t/y] \vdash \Delta}{x :_{P \oplus^* Q} X \mid \Gamma, \forall(y :_q A).\, \varphi \vdash \Delta}\ \forall_{\mathsf{L}}$$

First two are sound by representability, the other two are sound because they can be derived from SUBST + inverses of the first, which are sound.

The calculus: multiplicatives

$$\frac{z :_R Z;\ x :_P X \mid \Gamma \vdash \Delta, A \quad \otimes \quad z :_S Z;\ y :_Q Y \mid \Gamma' \vdash B, \Delta'}{z :_{R \oplus^* S} Z;\ x :_P X;\ y :_Q Y \mid \Gamma, \Gamma' \vdash \Delta, A \otimes B, \Delta'}\ \otimes_{\mathsf{R}}$$

$$\frac{z :_R Z;\ x :_P X \mid \Gamma, A \vdash \Delta \quad \otimes \quad z :_S Z;\ y :_Q Y \mid B, \Gamma' \vdash \Delta'}{z :_{R \oplus^* S} Z;\ x :_P X;\ y :_Q Y \mid \Gamma, A \otimes^* B, \Gamma' \vdash \Delta, \Delta'}\ \otimes^*_{\mathsf{L}}$$

shared variables add their grades, the others are juxtaposed

$$\frac{x :_P X\ \mathsf{ctx}}{x :_P X \mid \mathbf{1} \vdash}\ \mathbf{1}_{\mathsf{L}} \qquad \frac{x :_P X\ \mathsf{ctx}}{x :_P X \mid \vdash \mathbf{1}}\ \mathbf{1}_{\mathsf{R}}$$

$$\frac{x :_P X \mid \Gamma, A, B \vdash \Delta}{x :_P X \mid \Gamma, A \otimes B \vdash \Delta}\ \otimes_{\mathsf{L}} \qquad \frac{x :_P X \mid \Gamma \vdash A, B, \Delta}{x :_P X \mid \Gamma \vdash A \otimes^* B, \Delta}\ \otimes^*_{\mathsf{R}}$$

Sound by generalised Hölder

The calculus: additives

$$\frac{\infty}{x :_P X \mid \Gamma, \bot \vdash \Delta}\ \bot_{\mathsf{L}} \qquad \frac{\infty}{x :_P X \mid \Gamma \vdash \top, \Delta}\ \top_{\mathsf{R}}$$

$$(q \le P)\ \frac{x :_P X \mid \Gamma \vdash A, \Delta \quad \oplus^{-q} \quad x :_P X \mid \Gamma \vdash B, \Delta}{x :_P X \mid \Gamma \vdash A \wedge^q B, \Delta}\ \wedge^q_{\mathsf{R}}$$

$$\frac{x :_P X \mid \Gamma, A \vdash \Delta \quad \oplus^{q} \quad x :_P X \mid \Gamma, B \vdash \Delta}{x :_P X \mid \Gamma, A \wedge^q B \vdash \Delta}\ \wedge^q_{\mathsf{L}}$$

dually, $\vee^q_{\mathsf{L}}$ carries the guard $q \le P$, $\vee^q_{\mathsf{R}}$ has none

Sound by Minkowski's inequality.

Open, or not clear yet

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4. **hauptsatz**!

References I

[1] A. Benedek and R. Panzone, "The spaces {L^P}, with mixed norm", *Duke Mathematical Journal*, vol. 28, no. 3, pp. 301–324, Sep. 1961, ISSN: 0012-7094, 1547-7398. DOI: 10.1215/S0012-7094-61-02828-9. Accessed: Oct. 31, 2024. [Online]. Available: https://projecteuclid.org/journals/duke-mathematical-journal/volume-28/issue-3/The-space-LP-with-mixed-norm/10.1215/S0012-7094-61-02828-9.full.

[2] M. Capucci, "On Quantifiers for Quantitative Reasoning", arXiv: 2406.04936 [cs, math], Accessed: Jun. 26, 2024. [Online]. Available: http://arxiv.org/abs/2406.04936, pre-published.

[3] D. S. Cook and I. Weiss, "Diagrams of quantales and Lipschitz norms", *Fuzzy Sets and Systems*, vol. 444, pp. 79–102, 2022. Accessed: Aug. 27, 2025. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0165011421004267?casa_token=aaiqeFLjhaEAAAAA:hfr1l_4N8kFj25gW0ouj0Yz2pIwSOXxxo2cCXZMJN7-msJJ3LRQzZnDc8W3XM5F7wAv3lJL1.

References II

[4] F. Dagnino and F. Pasquali, "Logical Foundations of Quantitative Equality", in *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science*, ser. LICS '22, New York, NY, USA: Association for Computing Machinery, Aug. 4, 2022, pp. 1–13, ISBN: 978-1-4503-9351-5. DOI: 10.1145/3531130.3533337. Accessed: Mar. 3, 2023. [Online]. Available: https://doi.org/10.1145/3531130.3533337.

[5] F. W. Lawvere, "Metric spaces, generalized logic, and closed categories", *Rendiconti del seminario matématico e fisico di Milano*, vol. 43, pp. 135–166, 1973. DOI: 10.1007/BF02924844.

[6] D. Orchard, V.-B. Liepelt, and H. Eades III, "Quantitative program reasoning with graded modal types", *Proceedings of the ACM on Programming Languages*, vol. 3, no. ICFP, pp. 1–30, Jul. 2019. DOI: 10.1145/3341714.

[7] T. Petricek, D. Orchard, and A. Mycroft, "Coeffects: A calculus of context-dependent computation", in *Proceedings of the 19th ACM SIGPLAN International Conference on Functional Programming*, ser. ICFP '14, New York, NY, USA: Association for Computing Machinery, Aug. 19, 2014, pp. 123–135, ISBN: 978-1-4503-2873-9. DOI: 10.1145/2628136.2628160. Accessed: Oct. 15, 2026. [Online]. Available: https://dl.acm.org/doi/10.1145/2628136.2628160.

References III

[8] F. J. Valverde-Albacete and C. Peláez-Moreno, "The Case for Shifting the Rényi Entropy", *Entropy*, vol. 21, no. 1, p. 46, Jan. 2019, ISSN: 1099-4300. DOI: 10.3390/e21010046. Accessed: Nov. 20, 2026. [Online]. Available: https://www.mdpi.com/1099-4300/21/1/46.

[9] J. Wood and R. Atkey, "A Framework for Substructural Type Systems", in *Programming Languages and Systems*, I. Sergey, Ed., Cham: Springer International Publishing, 2022, pp. 376–402, ISBN: 978-3-030-99336-8. DOI: 10.1007/978-3-030-99336-8_14.

[10] j. wunder, A. A. de Amorim, P. Baillot, and M. Gaboardi, "Bunched Fuzz: Sensitivity for Vector Metrics", in *Programming Languages and Systems*, T. Wies, Ed., Cham: Springer Nature Switzerland, 2023, pp. 451–478, ISBN: 978-3-031-30044-8. DOI: 10.1007/978-3-031-30044-8_17.