

Adequate Losses via Quantitative Linear Logic

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joint work with Matteo Capucci, Robert Atkey, Charles Grellois and Matthew Daggitt

University of Southampton and Heriot-Watt University

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Outline

01 How I came to study quantitative logics

02 Matteo's work before QLL

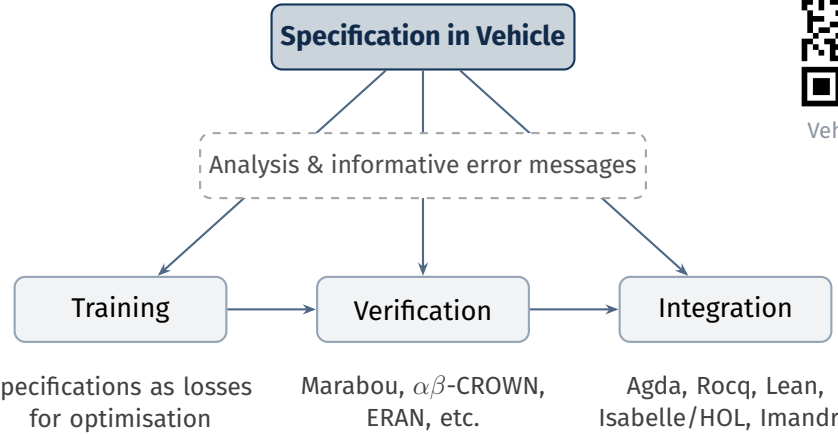
03 Propositional QLL

04 *p*QLL in Vehicle

05 What QLL promised, and what it delivered

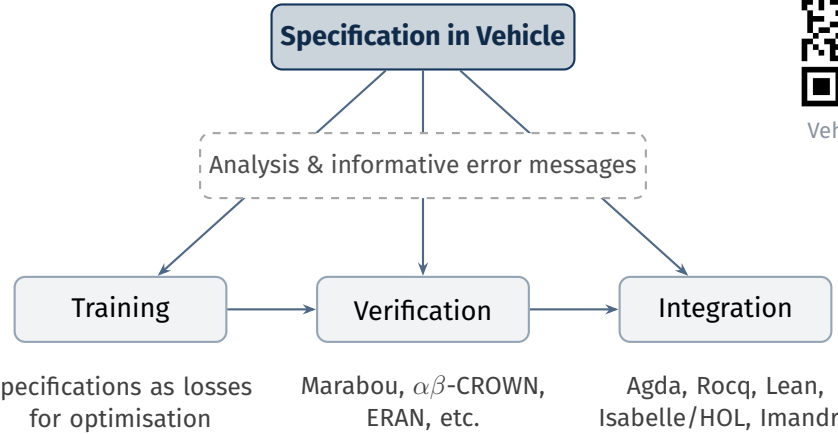
01 **How I came to study quantitative logics**
02 Matteo's work before QLL
03 Propositional QLL
04 *p*QLL in Vehicle
05 What QLL promised, and what it delivered

Vehicle



Vehicle tutorial

Vehicle

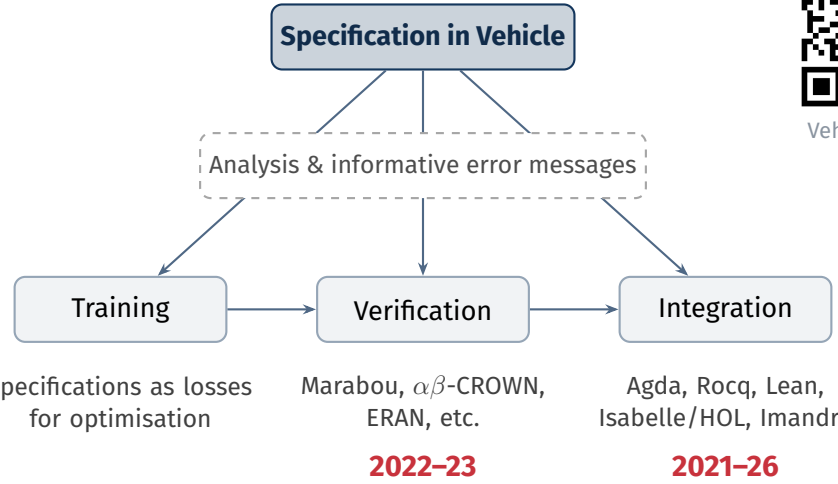


Vehicle tutorial

2021–26

M. L. Daggitt, E. Komendantskaya, A. Sirman, A. Bruni, S. Teuber, J. Smart, G. O. Passmore: Compositional Neural-Cyber-Physical System Verification in the Interactive Theorem Prover of Your Choice. Proc. ACM Program. Lang. 10(ICFP): 204–235 (2026).

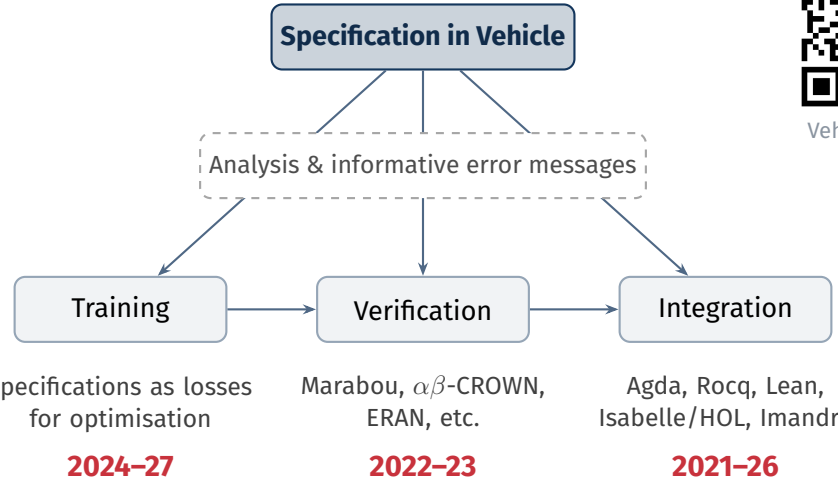
Vehicle



Vehicle tutorial

M. L. Daggitt, R. Atkey, W. Kokke, E. Komendantskaya, L. Arnaboldi: Compiling Higher-Order Specifications to SMT Solvers: How to Deal with Rejection Constructively. CPP 2023: 102–120.

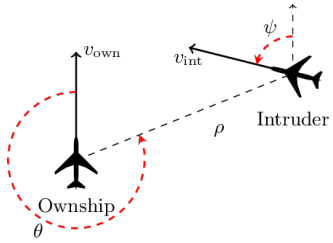
Vehicle



Vehicle tutorial

M. Capucci, R. Atkey, C. Grellois, M. L. Daggitt, E. Komendantskaya: Adequate Losses via Quantitative Linear Logic. POPL 2027.

ACAS Xu neural networks

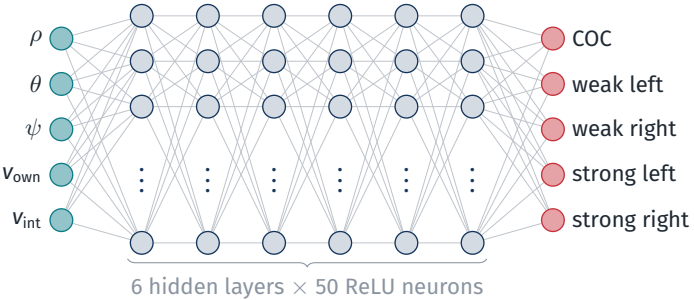


Inputs:

- Distance to intruder, ρ
- Angle to intruder, θ
- Intruder heading, ψ
- Speed, v_{own}
- Intruder speed, v_{int}

Outputs:

- Clear of conflict (COC)
- Weak left
- Weak right
- Strong left
- Strong right



ACAS Xu properties

Ten domain-specific properties and 45 networks in total.

ACAS Xu: Property 3

If the intruder is directly ahead and is moving towards the ownship, the score for COC will not be minimal.

$$1500 \leq \rho \leq 1800 \ \wedge -0.06 \leq \theta \leq 0.06 \ \wedge \psi \geq 3.10 \ \wedge v_{\text{own}} \geq 980 \ \wedge v_{\text{int}} \geq 960$$
$$\Rightarrow$$
$$\exists a \in \{\text{SL}, \text{L}, \text{R}, \text{SR}\}. f(\rho, \theta, \psi, v_{\text{own}}, v_{\text{int}})_a < f(\rho, \theta, \psi, v_{\text{own}}, v_{\text{int}})_{\text{COC}}$$

Generally

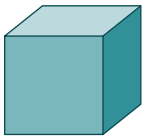
General property shape

$\forall \vec{x}.\, P(\vec{x}) \Rightarrow Q(f(\vec{x}))$

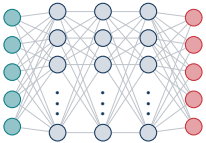
where $P(\vec{x})$ are linear inequalities on the input variables $\vec{x}$ and $Q(f(\vec{x}))$ are linear inequalities on the outputs $f(\vec{x})$.

Property 3: the cube $P(\vec{x})$

$$
\begin{aligned}
1500 \leq\ & \rho & \leq 1800 \\
-0.06 \leq\ & \theta & \leq 0.06 \\
3.10 \leq\ & \psi & \leq \pi \\
980 \leq\ & v_{\text{own}} & \leq 1200 \\
960 \leq\ & v_{\text{int}} & \leq 1200
\end{aligned}
$$



5 dimensions



ACAS Xu network f



all outputs $f(\vec{x})$ of the cube

How to interpret "Q: COC is not the minimal score for f"?

Differentiable logics: from a property to a loss

Differentiable logic

Interprets the postcondition $Q(f(\vec{x}))$ as a function on reals $[\![Q(f(\vec{x}))]\!]$:

Logic	Year	Domain	$[\![\top]\!]$	$[\![\bot]\!]$	$[\![\neg x]\!]$	$[\![x \wedge y]\!]$	$[\![x \vee y]\!]$
Gödel	1932	$[0,1]$	1	0	$1-x$	$\min\{x,y\}$	$\max\{x,y\}$
Product	1996	$[0,1]$	1	0	$1-x$	xy	$x+y-xy$
DL2	2019	$[0,\infty)$	0	∞	undefined	$x+y$	xy
QLL	2026	$[-\infty,\infty]$	$-\infty$	∞	$-x$	$\frac{1}{p}\log(e^{px}+e^{py})$	$-\frac{1}{p}\log(e^{-px}+e^{-py})$

M. Capucci, R. Atkey, C. Grellois, M. L. Daggitt, E. Komendantskaya: Adequate Losses via Quantitative Linear Logic. POPL 2027.

N. Ślusarz: A Unifying Framework for Differentiable Logics Using Interactive Theorem Proving. PhD thesis, Heriot-Watt University, Edinburgh, 2026.

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ACAS Xu, Property 3, in QLL (smaller is truer)

$$[\![\exists a \neq \mathrm{COC}.\, f(\vec{x})_a < f(\vec{x})_{\mathrm{COC}}]\!] \;=\; \bigvee_p \left\{ f(\vec{x})_a < f(\vec{x})_{\mathrm{COC}} \;\middle|\; a \in \{\mathrm{SL}, \mathrm{L}, \mathrm{R}, \mathrm{SR}\} \right\}$$

M. Capucci, R. Atkey, C. Grellois, M. L. Daggitt, E. Komendantskaya: Adequate Losses via Quantitative Linear Logic. POPL 2027.

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Differentiable logics: logical and analytic features

	Feature	Fuzzy				DL2	STL
		Gödel	Łukas.	Product	Yager		
Logical	Distinct non-linear/linear connectives	✗	✓	✓	✓	✗	✗
	Negation	✓	✓	✓	✓	✓[a]	✓
	Implication	✓	✓	✓	✓	✗	✓
	Good algebraic properties	✓	✓	✓	✓	✗	✗
Analytic[c]	Shadow-lifting connectives[d]	✗	✗	✓	✗	✓	✓
	$\wedge$, $\vee$ converge to min, max	✗[b]	✗[b]	✗[b]	✓	✗[b]	✓
	Non-vanishing gradients	✗	✗	✗	✗	✗	✓

[a] Only in the syntax, not as a semantic operation. [b] No varying family of connectives.

[c] For the connectives chosen for training. [d] On non-extremal values.

T. Flinkow, E. Komendantskaya, M. Capucci, R. Monahan: Quantitative Linear Logic for Neuro-Symbolic Learning and Verification. arXiv:2605.13845, 2026.

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The multiplicative reals $[0, \infty]_\otimes$

$a \otimes b$	0	$(0,\infty)$	∞
0	0	0	0
$(0,\infty)$	0	ab	∞
∞	0	∞	∞

conjunctive multiplication

a	0	$(0,\infty)$	∞
a^*	∞	$1/a$	0

duality: inversion

$a \multimap b$	0	$(0,\infty)$	∞
0	∞	0	0
$(0,\infty)$	∞	b/a	0
∞	∞	∞	∞

implication $a \multimap b$ (rows: b)

- Ordered by $\leq$: **bigger is truer**; 0 is false, ∞ is true, the unit is 1
- $*$-autonomous: $a \otimes b \leq c^* \iff a \leq (b \otimes c)^*$; residuated: $a \multimap b = a^* \otimes^* b$; isomix: $1^* = 1$
- Through $-\log$, the part $[1, \infty]$ is the **Lawvere quantale**

M. Capucci: On Quantifiers for Quantitative Reasoning. arXiv:2406.04936, 2024.

Soft additives: p-sums and harmonic p-sums

$$0 < p < \infty : \qquad a \oplus^p b \;:=\; (a^p + b^p)^{1/p} \qquad\qquad a \oplus^{-p} b \;:=\; (a^{-p} + b^{-p})^{-1/p}$$

	polarity	additive		multiplicative
duality $a^* := 1/a$	**positive**	$\bot := 0$ $a \oplus^p b$	$\xrightarrow[p\to\infty]{}$ $\bot := 0$ $a \oplus^\infty b := a \vee b$	$\mathbf{1} := 1$ $a \otimes b$
	negative	$\top := \infty$ $a \oplus^{-p} b$	$\xrightarrow[p\to\infty]{}$ $\top := \infty$ $a \oplus^{-\infty} b := a \wedge b$	$\mathbf{1} := 1$ $a \otimes^* b$

Red: disjunctive. Blue: conjunctive. p is the **hardness**: $a \oplus^{-p} b \;\le\; a \wedge b \;\le\; a \vee b \;\le\; a \oplus^p b$

Multiplication distributes over p-sums

$$a \otimes (b \oplus^p c) = (a \otimes b) \oplus^p (a \otimes c) \qquad\qquad a \otimes^* (b \oplus^{-p} c) = (a \otimes^* b) \oplus^{-p} (a \otimes^* c)$$

Napier's isomorphism: the additive reals

$$[0,\infty]_{\oplus^p,\otimes} \underset{-\log}{\overset{1/\exp}{\underset{\sim}{\longleftrightarrow}}} [-\infty,+\infty]_{\cup^p,+}$$

order-reversing: smaller is truer

	polarity	additive		multiplicative
duality $a^* := -a$	**positive**	$\bot := +\infty$ $a \cup^p b$	$\xrightarrow{p\to\infty}$ $\bot := +\infty$ $a \cup^\infty b := \min(a,b)$	$\mathbf{1} := 0$ $a + b$
	negative	$\top := -\infty$ $a \cap^p b$	$\xrightarrow{p\to\infty}$ $\top := -\infty$ $a \cap^\infty b := \max(a,b)$	$\mathbf{1} := 0$ $a +^- b$

$$a \cup^p b = -\tfrac{1}{p}\log\left(e^{-pa} + e^{-pb}\right) \qquad a \cap^p b = \tfrac{1}{p}\log\left(e^{pa} + e^{pb}\right) \qquad a \multimap b = b - a$$

Multiplicative: probabilities, likelihoods, ratios. **Additive**: logits, entropy, energy, losses.

Matteo's idea: quantifiers are integrals

- Sums behave like quantifiers. Compare $\forall i.\, (b_i \to a) \;=\; (\exists i.\, b_i) \to a$ with

$$\oplus_i^{-p}\, (b_i \multimap a) \;=\; (\oplus_i^{p}\; b_i) \multimap a$$

- But a sum grows with the size of its domain. Normalise it: a p**-mean**, over a probability space (I, di)

Soft quantifiers, bounded by a probability space

$$[\![\exists^p (i \in I).\, \varphi]\!] := \Big(\int_{i\in I} [\![\varphi]\!](i)^p \, di\Big)^{1/p} \qquad [\![\forall^p (i \in I).\, \varphi]\!] := \Big(\int_{i\in I} [\![\varphi]\!](i)^{-p} \, di\Big)^{-1/p}$$

$$\bigwedge \;\leq\; \forall^p \;\leq\; \exists^p \;\leq\; \bigvee \qquad \text{and the gap closes as } p \to \infty$$

M. Capucci: On Quantifiers for Quantitative Reasoning. arXiv:2406.04936, 2024.

Matteo's spectrum of sums and of means

Varying p: sum ($p = 1$), harmonic sum ($p = -1$), $\bigwedge$ ($p = -\infty$) and $\bigvee$ ($p = +\infty$)

$$\begin{array}{ccccccccccccc} \bigwedge & & \cdots & & \oplus^{-p} & & \cdots & & \oplus^{*} & \not\exists & \oplus & & \cdots & & \oplus^{p} & & \cdots & & \bigvee \\ -\infty & \longleftarrow & & \longleftarrow & -p & \longleftarrow & & \longleftarrow & -1 & \longleftarrow & 0 & \longleftarrow & 1 & \longleftarrow & p & \longrightarrow & & \longrightarrow & \infty \end{array} \tag{3.10}$$

The spectrum of means:

$$\begin{array}{ccccccccccccc} \bigwedge & & \cdots & & \int^{-p} & & \cdots & & \int^{*} & \int^{+0} \int^{-0} & \int & & \cdots & & \int^{p} & & \cdots & & \bigvee \\ -\infty & \longleftarrow & & \longleftarrow & -p & \longleftarrow & & \longleftarrow & -1 & \longleftarrow & 0 & \longleftarrow & 1 & \longleftarrow & p & \longrightarrow & & \longrightarrow & \infty \end{array} \tag{3.14}$$

M. Capucci: On Quantifiers for Quantitative Reasoning. arXiv:2406.04936, 2024.

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Quantitative sequent calculi

$$\mathcal{H} \;::=\; \Gamma \vdash \Delta \;\mid\; r \;\mid\; \mathcal{H} \otimes \mathcal{H} \;\mid\; \mathcal{H} \otimes^{*} \mathcal{H} \;\mid\; \mathcal{H} \oplus^{p} \mathcal{H} \;\mid\; \mathcal{H} \oplus^{-p} \mathcal{H}$$

- The **validity** $|\pi| \in [0, \infty]$ of a proof π: evaluate the constants at its leaves with the operations of its rules
- The **provability** of a sequent: $|\Gamma \vdash \Delta| \;:=\; \bigvee \{\, |\pi| \;\mid\; \pi \text{ a proof of } \Gamma \vdash \Delta \,\}$
- Qualitatively provable: $|\Gamma \vdash \Delta| > 0$

Quantity in the judgement

Not only in the models: after Lawvere (1973) and Pavelka (1979).

Sequents and hypersequents

The same definition with truth values (**Prop**, $\wedge$) and (**Prop**, $\wedge, \vee$).

*p*QLL: structural and multiplicative rules

$$\frac{1}{A \vdash A}\ \mathrm{AX} \qquad \frac{0}{\Gamma \vdash \Delta}\ \mathrm{EFQ} \qquad \frac{1}{\vdash \mathbf{1}}\ \mathbf{1}\mathrm{R} \qquad \frac{1}{\mathbf{1} \vdash}\ \mathbf{1}\mathrm{L}$$

$$\frac{\Gamma \vdash A, \Delta \quad \otimes \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \mathrm{CUT} \qquad \frac{\Gamma \vdash \Delta \quad \otimes \quad \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \mathrm{MIX}$$

$$\frac{\Gamma \vdash A, \Delta \quad \otimes \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'}\ \otimes\mathrm{R} \qquad \frac{\Gamma, A \vdash \Delta \quad \otimes \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \parr B \vdash \Delta, \Delta'}\ \parr\mathrm{L}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\ \otimes\mathrm{L} \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}\ \parr\mathrm{R} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^{\perp} \vdash \Delta}\ \perp\mathrm{L} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^{\perp}, \Delta}\ \perp\mathrm{R}$$

The rules of MALL (isomix: one unit **1** for $\otimes$ and $\parr$), with multiplication between the premises.

pQLL: additive rules

$$\frac{\Gamma \vdash A, \Delta \quad \oplus^{p} \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \vee^{p} B, \Delta} \vee^{p}\mathsf{R} \qquad \frac{\Gamma \vdash A, \Delta \quad \oplus^{-p} \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge^{p} B, \Delta} \wedge^{p}\mathsf{R}$$

$$\frac{\Gamma, A \vdash \Delta \quad \oplus^{-p} \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee^{p} B \vdash \Delta} \vee^{p}\mathsf{L} \qquad \frac{\Gamma, A \vdash \Delta \quad \oplus^{p} \quad \Gamma, B \vdash \Delta}{\Gamma, A \wedge^{p} B \vdash \Delta} \wedge^{p}\mathsf{L}$$

$$\frac{\infty}{\Gamma, \bot \vdash \Delta} \bot\mathsf{L} \qquad \frac{\infty}{\Gamma \vdash \top, \Delta} \top\mathsf{R}$$

- **Superposition**: to prove $A \vee^{p} B$ we never choose a branch; a proof of A and a proof of B are *summed*
- At $p = \infty$ the one-premise rules of MALL are interadmissible with these; for $p < \infty$ they are not

The validity of a proof: two examples

Idempotency has a price ($p = 1$)

$$\dfrac{\dfrac{}{A \vdash A}\,1\ \mathsf{AX} \qquad \oplus^{-1} \qquad \dfrac{}{A \vdash A}\,1\ \mathsf{AX}}{A \vee^{1} A \vdash A}\ \vee^{1}\mathsf{L}$$

validity $1 \oplus^{-1} 1 = \frac{1}{2}$

In general $2^{-1/p}$, which tends to 1 as $p \to \infty$.

EFQ is far from useless

$$\dfrac{\dfrac{}{A \vdash A}\,1\ \mathsf{AX} \qquad \oplus^{p} \qquad \dfrac{}{A \vdash B}\,0\ \mathsf{EFQ}}{A \vdash A \vee^{p} B}\ \vee^{p}\mathsf{R}$$

validity $1 \oplus^{p} 0 = 1$

A proof by EFQ has validity 0: the calculus stays consistent.

Cut elimination, and what follows from it

Theorem (cut elimination)

For every $0 < p \leq \infty$: $\quad |\Gamma \vdash \Delta|_{p\mathrm{QLL}} \;\leq\; |\Gamma \vdash \Delta|_{p\mathrm{QLL\ without\ CUT}}$.
Eliminating CUT from a proof is effective, and **never decreases its validity**.

- **MALL**: provable in (isomix) MALL $\iff |\Gamma \vdash \Delta|_{p\mathrm{QLL}} > 0 \iff |\Gamma \vdash \Delta|_{\infty\mathrm{QLL}} \geq 1$
- **Approximation**: $|\Gamma \vdash \Delta|_{p\mathrm{QLL}} \to |\Gamma \vdash \Delta|_{\infty\mathrm{QLL}}$ as $p \to \infty$
- Formalised in Rocq, on top of Laurent's YALLA (Janis Bailitis's talk, tomorrow)

J. Bailitis, R. Affeldt, A. Bruni, A. Coltellacci, E. Komendantskaya, K. Stark: Towards Quantitative Logics in Rocq. The Rocqshop 2026.

Softales: the algebraic semantics

A p-softale: a $[0,\infty]_\otimes$-enriched preorder, $\sqsubseteq : S \times S \to [0,\infty]$, with

$$1 \leq (a \sqsubseteq a) \qquad (a \sqsubseteq b) \otimes (b \sqsubseteq c) \;\leq\; (a \sqsubseteq c)$$

Unit 1, tensor $\bullet$, duality $(-)^\perp$

$$\begin{aligned} (a \sqsubseteq b) \otimes (c \sqsubseteq d) &\leq (a \bullet c \sqsubseteq b \bullet d) \\ (a \sqsubseteq b) &= (b^\perp \sqsubseteq a^\perp) \\ (a \bullet b \sqsubseteq c^\perp) &= (a \sqsubseteq (b \bullet c)^\perp) \end{aligned}$$

Bottom $\perp$ and p-joins $a \vee^p b$

$$\begin{aligned} (\perp \sqsubseteq a) &= \infty \\ (a \sqsubseteq c) \oplus^{-p} (b \sqsubseteq c) &\leq (a \vee^p b \sqsubseteq c) \\ (c \sqsubseteq a) \oplus^{p} (c \sqsubseteq b) &\leq (c \sqsubseteq a \vee^p b) \end{aligned}$$

- Generalises Lawvere's metric spaces and residuated lattices; e.g. $[0,\infty]$ with $\multimap$, and $[0,\infty]^A$

Soundness

In a p-softale S, a sequent $\Gamma \vdash \Delta$ is read as $\Gamma^\bullet \sqsubseteq \Delta^\circ$ (tensor of Γ, par of Δ), and a structure is evaluated with its operations.

Proposition (soundness)

For every rule $\dfrac{\mathcal{H}}{\mathcal{H}'}$ of pQLL and every p-softale S: $\quad |\mathcal{H}|_S \leq |\mathcal{H}'|_S$

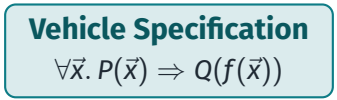
rule	law of softales
AX	reflexivity
CUT	transitivity
MIX, $\otimes$R	$\bullet$ is monotone
$\vee^p$L	minimality of the p-join
$\vee^p$R	majorisation by the p-join

- Formulae ordered by provability form the **classifying softale**; it is free on the atoms
- Hence **completeness**:

$$|\varphi \vdash \psi| = \bigwedge_S \bigwedge_{v : V \to S} (\llbracket \varphi \rrbracket_v \sqsubseteq \llbracket \psi \rrbracket_v)$$

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The ML backend: specification-driven training



Standard training
e.g. cross-entropy

Specification-driven training

$$\theta^* = \arg\min_{\theta} \mathop{\mathbb{E}}_{(\vec{x},\vec{y})\sim\mathcal{D}} \Big[\alpha\, \mathcal{L}(\vec{x},\vec{y};f_\theta) + (1-\alpha) \max_{\vec{x}' \in (\!|P(\vec{x})|\!)} [\![Q(f(\vec{x}))]\!](\vec{x},\vec{x}',\vec{y};f_\theta) \Big]$$

$(\!|P(\vec{x})|\!)$: the inputs that satisfy P $\quad$ $[\![Q(f(\vec{x}))]\!]$: the loss of Q, given by a differentiable logic

ACAS Xu $N_{1,8}$: Marabou verdicts, before and after training with QLL

Property	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}
Base model	✓ 4 s	✓ 5 min	✗ 2 s	✗ 1 s	✗ 1 s	t/o 2.5 h	t/o 2.5 h	t/o 2.5 h	✗ 1 s	✓ 2 min
One tour 10 epochs	✓ 3 s	✓ 20 s	✓ 1 s	✗ 2 s	✗ 1 s	✗ 2 min	t/o 15 min	t/o 15 min	✗ 2 s	✓ 19 min
Three tours 30 epochs	✓ 24 s	✓ 25 s	✓ 2 s	✓ 1 s	✗ 24 s	✗ 2 h	✗ 2 s	✗ 4 s	✓ 28 s	✓ 49 s

✓ verified ✗ falsified, with a counterexample t/o undecided within the time limit

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What QLL promised, and what it delivered

The promise	The delivery

What QLL promised, and what it delivered

The promise		The delivery
Connectives fit for gradient descent, yet still a logic	✓	p-sums: smooth, associative, no shadowing; idempotent in the limit $p \to \infty$

Differentiable logics: logical and analytic features

	Feature	Fuzzy				DL2	STL	QLL
		Gödel	Łukas.	Product	Yager			
Logical	Distinct non-linear/linear connectives	✗	✓	✓	✓	✗	✗	✓
	Negation	✓	✓	✓	✓	✓[a]	✓	✓
	Implication	✓	✓	✓	✓	✗	✓	✓
	Good algebraic properties	✓	✓	✓	✓	✗	✗	✓
Analytic[c]	Shadow-lifting connectives[d]	✗	✗	✓	✗	✓	✓	✓
	$\wedge$, $\vee$ converge to min, max	✗[b]	✗[b]	✗[b]	✓	✗[b]	✓	✓
	Non-vanishing gradients	✗	✗	✗	✗	✗	✓	✓

[a] Only in the syntax, not as a semantic operation. [b] No varying family of connectives.

[c] For the connectives chosen for training. [d] On non-extremal values.

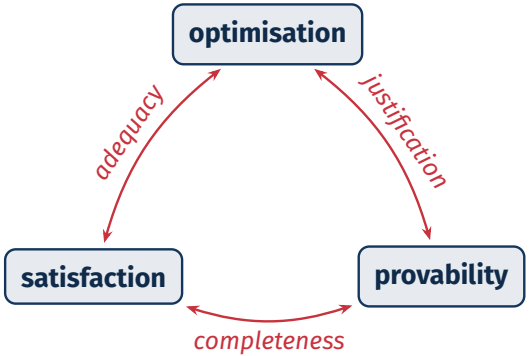
What QLL promised, and what it delivered

The promise		The delivery
Connectives fit for gradient descent, yet still a logic	✓	p-sums: smooth, associative, no shadowing; idempotent in the limit $p \to \infty$
A proof theory for quantities	✓	pQLL: cut elimination, decidability; MALL at $p = \infty$ and in the limit

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The promise: three things a differentiable logic must reconcile



- **Optimisation**: gradient descent on the loss $[\![\varphi]\!]_p(\vec{x}, \vec{x}', \vec{y}; f_\theta)$
- **Satisfaction**: $\vec{x}, \vec{x}', \vec{y}, f_\theta \models^p \varphi$
- **Provability**: $\vdash \varphi|_{\mathbb{T}_{\vec{x}', \vec{y}, f_\theta}}$ in a proof calculus

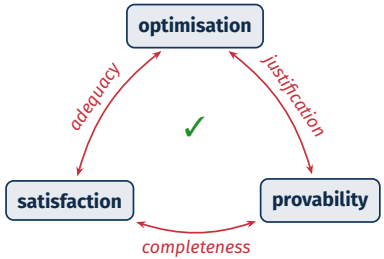
M. Capucci, R. Atkey, C. Grellois, M. L. Daggitt, E. Komendantskaya: Adequate Losses via Quantitative Linear Logic. POPL 2027. arXiv:2605.13348.

Delivered: loss is satisfaction is provability

Theorem (quantitative adequacy), for every hardness $0 < p \leq \infty$

$$\underbrace{e^{-[\![\varphi]\!]_p}}_{\text{the loss}} = \underbrace{\vec{x}, \vec{x}', \vec{y}, \boxed{f_\theta} \models^p \varphi}_{\text{quantitative satisfaction}} = \underbrace{|\vdash \varphi|_{\mathbb{T}_{\vec{x}', \vec{y}, \boxed{f_\theta}}}}_{\text{provability}}$$

- **Every** value of the loss has a logical reading, not only the extremal ones
- Minimising the loss **increases** the provability of the specification
- Soft p for training, $p = \infty$ for testing
- Qualitatively: $\boxed{f_\theta}$ satisfies φ iff $|\vdash \varphi|_{\mathbb{T}_{\vec{x}', \vec{y}, \boxed{f_\theta}}} > 0$



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Adequacy: a lower loss means a better satisfied specification	✓	loss = satisfaction = provability, for *every* value of the loss
Better training in practice	✓	Networks trained on the pQLL loss verify more often: Flinkow et al.; the ICFEM case studies

T. Flinkow, E. Komendantskaya, M. Capucci, R. Monahan: Quantitative Linear Logic for Neuro-Symbolic Learning and Verification. arXiv:2605.13845, 2026.

E. Komendantskaya, M. Casadio, J. Ciupa, T. Flinkow, J. Hall, C. Kessler, A. Sirman: Specification Is All You Need. ICFEM 2026.

What QLL promised, and what it delivered

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Better training in practice	✓	Networks trained on the pQLL loss verify more often: Flinkow et al.; the ICFEM case studies
Quantifiers: $\forall x$, sup, $\mathbb{E}$	$\sim$	Not yet: the quantifier is searched for outside the logic. Next: first-order pQLL

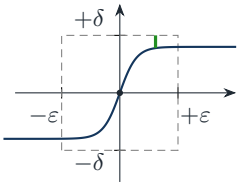
Possible additional slides

Problem 1: a yes-or-no satisfaction cannot be adequate

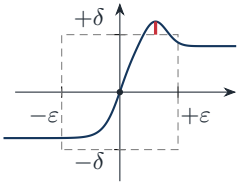
Robustness around a data point $\hat{x}$:

$$\forall x.\ |x - \hat{x}| \leq \varepsilon \Rightarrow |f_\theta(x) - f_\theta(\hat{x})| \leq \delta$$

- DL2: $[\![\varphi]\!] = 0$ iff f_θ satisfies φ
- But the loss is never exactly 0
- And it does not explain why *lowering* the loss *improves* satisfaction



✓ satisfied



✗ violated

Thesis 1

The verification of machine learning models should be founded on a **quantitative** reconception of logic: quantitative proofs, quantitative judgements, quantitative satisfaction.

Problem 2: analysis against algebra

Gradient descent wants

- smooth connectives with $\partial_i \odot > 0$
- else *shadowing*: $\max(a, b)$ sends no gradient to the smaller argument

Proof theory wants

- additives and multiplicatives
- residuated lattices force the additives to be $\min$ and $\max$

No-go theorem (Várnai and Dimarogonas): no connective is at once idempotent, associative and shadow-lifting.

Thesis 2

On the reals there are obvious additive and multiplicative operations: **addition and multiplication themselves**. Use them, **without sacrificing the additives of the proof calculus**.

P. Várnai, D. V. Dimarogonas: On Robustness Metrics for Learning STL Tasks. ACC 2020.

Delivered: connectives fit for gradient descent

- **Naturality**: logits and cross-entropy are additive quantities, so the logical term of the loss must be one too
- Hence the additive semantics of pQLL, through Napier's isomorphism:

$$\text{and: } a \cap^p b = \frac{1}{p}\log\left(e^{pa} + e^{pb}\right) \qquad \text{or: } a \cup^p b = -\frac{1}{p}\log\left(e^{-pa} + e^{-pb}\right)$$

  These are the connectives behind the **QLL column** of the feature table (slide 23), and Vehicle's `qllAdditive`

- Smooth, with partial derivatives that never vanish: $\partial_a\,(a \cap^p b) = \dfrac{e^{pa}}{e^{pa} + e^{pb}}$
- Idempotency is lost, at the price of $\frac{1}{p}\log 2$, which vanishes as $p \to \infty$

Associative, shadow-lifting, and idempotent in the limit: the no-go theorem is circumvented.

Grounded completeness: the reals are enough

- pQLL is complete for all softales, but not for the single model $[0, \infty]$
- Fix a valuation $v : V \to [0, \infty]$ of the atoms. The **grounded theory** $\mathbb{T}_v$ is pQLL* (that is, pQLL with MIX*) and

$$\frac{v(a)}{\vdash a} \qquad \frac{v(a)^*}{a \vdash} \qquad \text{for every atom } a$$

Theorem (grounded completeness)

For every formula φ, with $[\![-]\!]_v$ the unique model of $\mathbb{T}_v$ in $[0, \infty]$:

$$[\![\varphi]\!]_v \;=\; |\vdash \varphi|_{\mathbb{T}_v}$$

- The value of a formula in the reals *is* its provability
- MIX* is needed only when v takes the values 0 or ∞

The theory of a neuro-symbolic specification

- A property $\forall x.\ a \le x \le b \Rightarrow \varphi(x)$; the quantifier is discharged by adversarial search (PGD)
- What remains is a *propositional* φ, over atoms $\ulcorner y_i \urcorner$ (outputs of the network) and $\ulcorner \hat{y}_i \urcorner$ (labels)

The grounded theory $\mathbb{T}_{x,\hat{y},f_\theta}$: pQLL* and the grounding schema

$$\frac{e^{-f_\theta(x)_i}}{\vdash \ulcorner y_i \urcorner} \qquad \frac{e^{f_\theta(x)_i}}{\vdash \ulcorner y_i \urcorner^\perp} \qquad \frac{\hat{y}_i}{\vdash \ulcorner \hat{y}_i \urcorner} \qquad \frac{\hat{y}_i^*}{\vdash \ulcorner \hat{y}_i \urcorner^\perp}$$

The atoms are not statements *about* the network: they **are** the network's outputs, read logically.

Adequacy (1): the canonical model and quantitative satisfaction

- Every theory of neuro-symbolic specifications has a canonical interpretation in the additive reals $[-\infty, +\infty]_{\cup p,+}$, induced by the valuation of the atoms

$$[\![\ulcorner \hat{y}_i \urcorner]\!]_p = -\log \hat{y}_i, \qquad [\![\ulcorner y_i \urcorner]\!]_p = \boxed{f_\theta}(x)_i$$

- So for each $0 < p \leq \infty$ there is a model of the theory

$$\hat{x}, \hat{y}, x, \boxed{f_\theta} \models^p - \ : \ \mathbf{LT}^p\left[\mathbb{T}_{x,\hat{y},\boxed{f_\theta}}\right] \to [-\infty, +\infty]_{\cup p,+}$$

Definition (quantitative satisfaction)

The quantitative satisfaction of a specification φ by $\boxed{f_\theta}$ at $(\hat{x}, \hat{y})$ and $a \leq x \leq b$ is the semantic validity of $\vdash \varphi$ in the canonical interpretation:

$$\hat{x}, \hat{y}, x, \boxed{f_\theta} \models^p \varphi \ = \ \mathbf{1} \sqsubseteq [\![\varphi]\!]_p \ = \ e^{-[\![\varphi]\!]_p}$$

This is adequacy: every value of the loss $[\![\varphi]\!]_p$ is a degree of satisfaction by a model.

Adequacy (2): loss is satisfaction is provability

Theorem (logical adequacy of property-driven training)

Given f_θ, a data point $(\hat{x}, \hat{y}) \in \mathcal{D}$ and any $a \leq x \leq b$:

$$\hat{x}, \hat{y}, x, f_\theta \models^p \varphi \quad = \quad |\vdash \varphi|_{\mathbb{T}_{x,\hat{y},f_\theta}}$$

Proof: grounded completeness, applied to the theory $\mathbb{T}_{x,\hat{y},f_\theta}$.

- **Every** value of the loss has a logical reading, not only the extremal ones
- Minimising the loss **improves** satisfaction: $[\![\varphi]\!]_p$ decreases as $\hat{x}, \hat{y}, x, f_\theta \models^p \varphi$ increases
- It holds for **every** p: the soft semantics at training time, and the hard one at test time, where $\hat{x}, \hat{y}, x, f_\theta \models^\infty \varphi$ has a traditional logical reading
- Binary satisfaction is recovered, through MALL: f_θ satisfies φ iff $|\vdash \varphi|_{\mathbb{T}_{x,\hat{y},f_\theta}} > 0$

Backup: example: Bayesian inference in 1QLL*

- Outcomes $\omega \in \Omega$, with probabilities $P(\omega)$: one atom each, *grounded* by axioms

$$\frac{P(\omega)}{\vdash \omega} \qquad \frac{P(\omega)^*}{\omega \vdash}$$

- An event $A \subseteq \Omega$ is the formula $\bigvee^1_{a \in A} a$
- One more rule, for grounded theories:

$$\frac{\Gamma_1 \vdash \Delta_1 \quad \otimes^* \quad \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \text{MIX}^*$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\bigoplus^1_{b \in A \cap B} \bigoplus^{-1}_{a \in A} P(a)^* \otimes^* P(b)}{\bigoplus^1_b \bigoplus^{-1}_a (a \vdash \otimes^* \vdash b)}\ \text{axioms}}{\bigoplus^1_b \bigoplus^{-1}_a a \vdash b}\ \text{MIX}^*}{\bigoplus^1_b A \vdash b}\ \vee^1\text{L}}{A \vdash A \cap B}\ \vee^1\text{R}$$

Conditional probability is provability: $\quad |A \vdash A \cap B| \;=\; \dfrac{P(A \cap B)}{P(A)} \;=\; P(B \mid A)$

Backup: Cut elimination: the additive case

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \quad \oplus^p \quad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \vee^p B, \Delta}\, \vee^p \mathsf{R} \qquad \otimes \qquad \dfrac{\dfrac{\pi_3}{\Gamma', A \vdash \Delta'} \quad \oplus^{-p} \quad \dfrac{\pi_4}{\Gamma', B \vdash \Delta'}}{\Gamma', A \vee^p B \vdash \Delta'}\, \vee^p \mathsf{L}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, \mathsf{CUT}$$

- As in MALL: cut π_1 against π_3 (on A), or π_2 against π_4 (on B)
- New: the validity must not decrease. It does not, because in $[0, \infty]$

$$\big(|\pi_1| \oplus^p |\pi_2|\big) \otimes \big(|\pi_3| \oplus^{-p} |\pi_4|\big) \;\leq\; \big(|\pi_1| \otimes |\pi_3|\big) \vee \big(|\pi_2| \otimes |\pi_4|\big)$$

- So: keep the cut of maximal validity

Backup: *p*QLL and MALL

Theorems (conservativity): for a sequent $\Gamma \vdash \Delta$, the following are equivalent

1. $\Gamma \vdash \Delta$ is provable in (isomix) MALL
2. $|\Gamma \vdash \Delta|_{p\mathrm{QLL}} > 0$, for any hardness p — *additive mode of truth*
3. $|\Gamma \vdash \Delta|_{\infty\mathrm{QLL}} \geq 1$ — *multiplicative mode of truth*

Theorem (approximation)

$$|\Gamma \vdash \Delta|_{p\mathrm{QLL}} \longrightarrow |\Gamma \vdash \Delta|_{\infty\mathrm{QLL}} \qquad \text{as } p \to \infty$$

- Qualitatively, *p*QLL is the hypersequent calculus for MALL; EFQ plays the part of external weakening
- Soft connectives for training, hard ones for testing, and a continuous passage between them

Backup: Softales (1): tensor and duality

A $[0,\infty]_{\otimes}$-preorder (Lawvere 1973)

A set S with $\sqsubseteq : S \times S \to [0,\infty]$ such that

$$1 \leq (a \sqsubseteq a) \qquad (a \sqsubseteq b) \otimes (b \sqsubseteq c) \;\leq\; (a \sqsubseteq c)$$

Multiplicative structure: a unit 1, a tensor $\bullet$ and a duality $(-)^{\perp}$

$$\begin{aligned}
(a \sqsubseteq b) \otimes (c \sqsubseteq d) &\leq (a \bullet c \sqsubseteq b \bullet d) && && \bullet \text{ is monotone} \\
(a \sqsubseteq b) &= (b^{\perp} \sqsubseteq a^{\perp}), && a^{\perp\perp} = a, \quad \mathbf{1}^{\perp} = \mathbf{1} && \text{involutive duality} \\
(a \bullet b \sqsubseteq c^{\perp}) &= \left(a \sqsubseteq (b \bullet c)^{\perp}\right) && && *\text{-autonomy}
\end{aligned}$$

Backup: Softales (2): soft joins

Additive structure of a p-softale: a bottom $\bot$ and p-joins $a \vee^p b$

$$
\begin{aligned}
(\bot \sqsubseteq a) &= \infty \\
(a \sqsubseteq c) \oplus^{-p} (b \sqsubseteq c) &\leq (a \vee^p b \sqsubseteq c) && \text{minimality} \\
(c \sqsubseteq a) \oplus^{p} (c \sqsubseteq b) &\leq (c \sqsubseteq a \vee^p b) && \text{majorisation}
\end{aligned}
$$

- In a lattice, the two together are equivalent to $(a \leq c) \wedge (b \leq c) \iff (a \vee b \leq c)$
- Softly they are not: that needs $\wedge$ to distribute over $\vee$, and idempotence
- **Examples**: $[0, \infty]$ with $\multimap$; involutive commutative residuated lattices; functions $[0, \infty]^A$, pointwise
- The additive preorder of a p-softale, and the multiplicative preorder of an ∞-softale, are involutive residuated lattices

Backup: A reading: conversion rates

- Linear logic: $\Gamma \vdash A$ says that the resources Γ *can be converted* into A
- Quantitative linear logic: $|\Gamma \vdash A|$ is the **conversion rate** from Γ to A

Example, in 1QLL

$$|1\text{€} \vdash \text{Pizza}| = 5 \qquad |1\text{€} \vdash \text{Soda}| = 4 \qquad \xRightarrow{\vee^1_R} \qquad |1\text{€} \vdash \text{Pizza} \vee^1 \text{Soda}| = 9$$

- A conversion rate > 0 means convertible: this will be provability in MALL

Backup: why groundedness matters

Distributivity:

$$(A \otimes C) \wedge^p (A \otimes B) \vdash A \otimes (B \wedge^p C)$$

- Its proof of maximal validity ends in the structure $\left(C \vdash B \oplus^p 1\right) \oplus^{-p} \left(1 \oplus^p B \vdash C\right)$
- If neither $C \vdash B$ nor $B \vdash C$ has a proof of positive validity, this is 0
- If $|B \vdash C|^* \leq |C \vdash B|$, it is at least 1: distributivity holds
- That inequality holds in grounded pQLL*-theories, by MIX*